Weil representation

Heisenberg group

Over real numbers

$$H = H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

the group with respect to matrix multiplication

$$\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 & c_1 + c_2 + a_1 b_2 \\ 0 & 1 & b_1 + b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Schrödinger representation.

$H_3(\mathbb{R})$ acts on $L^2(\mathbb{R})$.

Fix $\hbar \in \mathbb{R} \setminus \{0\}$

$$\left(\pi_\hbar \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} f\right)(x) = e^{i\hbar c} \cdot e^{ibx} \cdot f(x + \hbar a)$$

Shifts

multiplications by e^{ixy}

$f \in L^2(\mathbb{R})$

Exercise: $\pi_\hbar \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ is a unitary operator on $L^2(\mathbb{R})$.

Check that $\pi_\hbar$ is a representation:

$$\pi_\hbar\left(\begin{pmatrix}1 & a_1 & c_1\\ 0 & 1 & b_1\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & a_2 & c_2\\ 0 & 1 & b_2\\ 0 & 0 & 1\end{pmatrix}\right)(f)(x) = \pi_\hbar\begin{pmatrix}1 & a_1+a_2 & c_1+c_2+a_1b_2\\ 0 & 1 & b_1+b_2\\ 0 & 0 & 1\end{pmatrix}(f)(x)$$

$$= e^{i\hbar(c_1+c_2+a_1b_2)}\cdot e^{i(b_1+b_2)x}\cdot f(x+\hbar(a_1+a_2))$$

$$\pi_\hbar\begin{pmatrix}1 & a_1 & c_1\\ 0 & 1 & b_1\\ 0 & 0 & 1\end{pmatrix}\left(\pi_\hbar\begin{pmatrix}1 & a_2 & c_2\\ 0 & 1 & b_2\\ 0 & 0 & 1\end{pmatrix}(f)\right)(x) = \pi_\hbar\begin{pmatrix}1 & a_1 & c_1\\ 0 & 1 & b_1\\ 0 & 0 & 1\end{pmatrix}\left(y\mapsto e^{i\hbar c_2}\cdot e^{ib_2y}\cdot f(y+\hbar a_2)\right)(x)$$

$$= e^{i\hbar c_1}\, e^{ib_1x}\cdot\left(e^{i\hbar c_2}\cdot e^{ib_2y} f(y+\hbar a_2)\Big|_{y=x+\hbar a_1}\right)$$

$$= e^{i\hbar(c_1+c_2+b_2a_1)}\cdot e^{i(b_1+b_2)y}\, f(x+\hbar(a_1+a_2)).$$

Theorem (Stone-von Neumann theorem)

Every strongly continous irreducible unitary representation of of H on $L_2(\mathbb{R})$ is equivalent to $\pi_\hbar$ for some $\hbar$.

The Structure of the Heisenberg group as the central extension

The center of $H = H_3(\mathbb{R})$ is

$$Z = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, c \in \mathbb{R} \right\}.$$

$Z \simeq (\mathbb{R}, +)$, $\gamma : \mathbb{R} \to Z$ $\quad \gamma(c) := \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Define $Q := H/Z$

$$Q = \left\{ \begin{pmatrix} 1 & a & * \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}. \quad Q \simeq \mathbb{R}^2$$

Short exact sequence: $\quad 0 \to Z \longrightarrow H \longrightarrow Q \longrightarrow 0$

Action of $SL_2(\mathbb{R})$ on $H_3(\mathbb{R})$.

The group $SL_2(\mathbb{R})$ acts on $\mathbb{R}^2$;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

$$\begin{tikzcd} H \arrow[d, "\pi"'] & \circlearrowleft SL_2(\mathbb{R}) \\ Q \arrow[u, bend right, dashed, "\sigma"'] & \circlearrowleft SL_2(\mathbb{R}) \end{tikzcd}$$

$$\sigma : Q \longrightarrow H \qquad (x,y) \longmapsto \begin{pmatrix} 1 & x & \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

⚠ σ is not a homomorphism.

$$\sigma(x_1,y_1)\cdot\sigma(x_2,y_2) = \begin{pmatrix} 1 & x_1 & \frac{1}{2}x_1y_1 \\ & 1 & y_1 \\ & & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & x_2 & \frac{1}{2}x_2y_2 \\ & 1 & y_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1+x_2 & \frac{1}{2}x_1y_1+\frac{1}{2}x_2y_2+x_1y_2 \\ & 1 & y_1+y_2 \\ & & 1 \end{pmatrix}$$

$$G(x_1,y_1)\cdot G(x_2,y_2)=\begin{pmatrix}1 & x_1 & \frac{1}{2}x_1y_1\\ & 1 & y_1\\ & & 1\end{pmatrix}\cdot\begin{pmatrix}1 & x_2 & \frac{1}{2}x_2y_2\\ & 1 & y_2\\ & & 1\end{pmatrix}=\begin{pmatrix}1 & x_1+x_2 & \frac{1}{2}x_1y_1+\frac{1}{2}x_2y_2+x_1y_2\\ & 1 & y_1+y_2\\ & & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & x_1+x_2 & \frac{1}{2}(x_1+x_2)(y_1+y_2)+\frac{1}{2}x_1y_2-\frac{1}{2}x_2y_1\\ & 1 & y_1+y_2\\ & & 1\end{pmatrix}=G(x_1+x_2,y_1+y_2)\cdot\begin{pmatrix}1 & 0 & \frac{1}{2}(x_1y_2-x_2y_1)\\ & 1 & 0\\ & & 1\end{pmatrix}$$

Belongs to $\mathcal{Z}$

$$\frac{1}{2}(x_1y_2-x_2y_1)=\frac{1}{2}\det\begin{pmatrix}x_1 & x_2\\ y_1 & y_2\end{pmatrix}\qquad (\star)$$

$$v = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in H \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

Exercise: Compute w.

$$v = \sigma(x,y) \cdot w \in Z$$

Set

$$g \cdot v := \sigma(\underbrace{g(x,y)}_{=(ax+by,\, cx+dy)}) \cdot w$$

Is this an action?

$$g_1 \cdot g_2 (v) = g_1 (g_2(x,y) \cdot w) = g_1 \cdot g_2(x,y) \cdot w$$

Is the action of g a group automorphism of H?

$$g(v_1 \cdot v_2) \overset{?}{=} g(v_1) \cdot g(v_2)$$

Suppose that $v_1 = \sigma(x_1, y_1) \cdot w_1$ $v_2 = \sigma(x_2, y_2) \cdot w_2$ for some $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in \mathbb{R}^2$ and $w_1, w_2 \in \mathcal{Z}$.

$$g(v_1 \cdot v_2) = g(\sigma(x_1, y_1) \cdot w_1 \cdot \sigma(x_2, y_2) \cdot w_2)$$

$$= g(\sigma(x_1, y_1)\sigma(x_2, y_2) \cdot w_1 \cdot w_2) =$$

$$= g\left(\sigma(x_1 + x_2, y_1 + y_2) \begin{pmatrix} 1 & 0 & \frac{1}{2}(x_1 y_2 - x_2 y_1) \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cdot w_1 \cdot w_2\right)$$

$$\boxed{\underline{\text{Notation}}: \varsigma : \mathbb{R} \to \mathcal{Z},\ \varsigma(c) = \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix}}$$

$$= \sigma(g(x_1 + x_2, y_1 + y_2)) \cdot \varsigma\left(\tfrac{1}{2}(x_1 y_2 - x_2 y_1)\right) \cdot w_1 \cdot w_2$$

$$g(v_1) \cdot g(v_2) = g(\sigma(x_1, y_1) \cdot w_1)\, g(\sigma(x_2, y_2) \cdot w_2)$$

$$= \sigma(g(x_1, y_1)) \cdot \sigma(g(x_2, y_2)) \cdot w_1 \cdot w_2$$

The action of g on H is a group homomorphism if

$$(\star)\quad G\left(g(x_1+x_2,\, y_1+y_2)\right)\cdot \zeta\left(\tfrac{1}{2}(x_1y_2-x_2y_1)\right) = G\left(g(x_1,y_1)\right)\cdot G\left(g(x_2,y_2)\right)$$

By $(\star)$:

$$G(g(x_1,y_1))\, G(g(x_2,y_2)) = G\left(g(x_1,y_1)+g(x_2,y_2)\right)\cdot \zeta\left(\tfrac{1}{2}(\tilde{x}_1\tilde{y}_2-\tilde{x}_2\tilde{y}_1)\right)$$

$$g(x_1,y_1)=(\tilde{x}_1,\tilde{y}_1) \qquad g(x_2,y_2)=(\tilde{x}_2,\tilde{y}_2)$$

$(\star)$ is equivalent to $x_1y_2-x_2y_1=\tilde{x}_1\tilde{y}_2-\tilde{x}_2\tilde{y}_1$.

$$\det\begin{pmatrix} x_1 & x_2\\ y_1 & y_2\end{pmatrix} = \det\begin{pmatrix}\tilde{x}_1 & \tilde{x}_2\\ \tilde{y}_1 & \tilde{y}_2\end{pmatrix}.$$

Thus the action of g is a group homomorphism of H $\blacksquare$

- $SL_2(\mathbb{R})$ acts on H
- Action of $SL_2(\mathbb{R})$ on Z is trivial
- H acts on $L_2(\mathbb{R})$
- Uniqueness & irreducibility of Schrödinger representation

Let $g \in SL_2(\mathbb{R})$

$$\pi_\hbar : H \to \mathcal{U}(L_2(\mathbb{R}))$$
$$\pi_\hbar \circ g : H \to \mathcal{U}(L_2(\mathbb{R}))$$

There exists $A_g \in GL(L_2(\mathbb{R}))$ s.t $\pi_\hbar \circ g = A_g \pi_\hbar A_g^{-1}$

A_g is unique up to scalar $\qquad \pi(gv) = A_g \pi(v) A_g^{-1}$

$g \mapsto A_g$ is a projective representation.

$$\rho : SL_2(\mathbb{R}) \longrightarrow PGL(L_2(\mathbb{R}))$$

$v \in H$

$g \in SL_2(\mathbb{R})$

$$\begin{tikzcd} L_2(\mathbb{R}) \arrow[r, "\pi_\hbar(v)"] \arrow[d, "A_g"'] & L_2(\mathbb{R}) \arrow[d, "A_g"] \\ L_2(\mathbb{R}) \arrow[r, "\pi_\hbar(g \cdot v)"'] & L_2(\mathbb{R}) \end{tikzcd}$$

Generators of $SL_2(\mathbb{R})$:

$$s := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad v(u) := \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \qquad d(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

$$A_s : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \qquad f \mapsto \hat{f}$$

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{i\hbar xy} \, dy$$

⚠ This not a unitary operator (only up to a constant)

$$\begin{tikzcd} L_2(\mathbb{R}) \arrow[r, "\pi_\hbar(v)"] \arrow[d, "A_s"'] & L_2(\mathbb{R}) \arrow[d, "A_s"] \\ L_2(\mathbb{R}) \arrow[r, "\pi_\hbar(g\cdot v)"'] & L_2(\mathbb{R}) \end{tikzcd}$$

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$s \cdot \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd} f \arrow[r] \arrow[d, "A_s"'] & f(x+\hbar a) \arrow[d, "A_s"] \\ g \arrow[r] & e^{-iax} g(x) \end{tikzcd}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -a \end{pmatrix}$$

$$\pi_\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} f(x) = e^{-iax} \cdot f(x)$$

$$\pi_\hbar \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f(x) = f(x+\hbar a)$$

Question!

Can we prove this statement by elementary methods?

$$A_{v(u)} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$$
$$f(x) \mapsto e^{-\frac{u}{2} i\hbar x^2} \cdot f(x)$$

(see computations on pages 20-23)

$$A_{d(t)} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$$
$$f(x) \mapsto f(t^{-1} x)$$

Let $f(x) = e^{\pi i x^2 \cdot \tau}$, $\tau \in \mathfrak{h}$

Fix $\hbar = \frac{1}{2\pi}$

Exercise! "$A_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \left(e^{\pi i x^2 \tau} \right) \doteq e^{\pi i x^2 \frac{-d\tau + c}{b\tau - a}}$"

the reason for this twist is the choice we have made in the definition of $\mathcal{J}_\hbar$

Heisenberg group over a finite field.

Let K be a field, $\mathrm{char}(K) \neq 2$

$$H = H_3(K) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in K \right\}$$

the group with respect to matrix product.

$$\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c|c} 1 & a_1+a_2 & c_1+c_2+a_1b_2 \\ \hline 0 & 1 & b_1+b_2 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Schrödinger representation.

$H_3(K)$ acts on $L^2(K)$. $(K=\mathbb{R},\ K=\mathbb{F}_p)$

Let $\psi : K \to \mathbb{C}^*$ be an irreducible character

Shifts

$$\left(\pi_\psi\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix} f\right)(x) = \psi(c-bx)\cdot f(x-a)$$

multiplications by ψ

$f \in L^2(K)$

Exercise! $\pi_\psi\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}$ is a unitary operator on $L^2(K)$.

For $K = \mathbb{F}_p$ we also have an analogue of Stone-von Neuman Thm.

$\boxed{p \text{ is odd}}$ $\quad \rho : SL_2(\mathbb{F}_p) \to PGL_p(\mathbb{C})$

$SL_2(\mathbb{F}_p)$ is generated by

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A_s : L_2(\mathbb{F}_p) \to L_2(\mathbb{F}_p), \quad f \mapsto \hat{f}$$

$$\hat{f}(y) = \sum_{x \in \mathbb{F}_p} \psi(y \cdot x) \cdot f(x)$$

$$A_t : L_2(\mathbb{F}_p) \to L_2(\mathbb{F}_p), \quad f(x) \mapsto \psi\left(-\tfrac{1}{2}x^2\right) f(x).$$

Remark:

$$\begin{tikzcd} \rho : SL_2(\mathbb{R}) \arrow[r] & PGL(L_2(\mathbb{R})) \\ \tilde{\rho} : Mp_2(\mathbb{R}) \arrow[r] \arrow[u] & GL(L_2(\mathbb{R})) \arrow[u] \end{tikzcd}$$

so called methaplectic group, double cover of $SL_2(\mathbb{R})$

p is odd

$$\begin{tikzcd} \rho : SL_2(\mathbb{F}_p) \arrow[r] & PGL_p(\mathbb{C}) \\ \tilde{\rho} : SL_2(\mathbb{F}_p) \arrow[r] \arrow[u, "id"] & GL_p(\mathbb{C}) \arrow[u] \end{tikzcd}$$

$\boxed{\text{Computations of } \rho\begin{pmatrix}1&0\\u&1\end{pmatrix}}$

Consider an element $A = \begin{pmatrix}1&a&0\\0&1&0\\0&0&1\end{pmatrix} \in H$

The Schrödinger represantation of A acts on $L_2(\mathbb{R})$ by

$$\pi_\hbar(A): \quad f(x) \mapsto f(x+\hbar a)$$

Our definition of the action of $SL_2(\mathbb{R})$ on H implies:

$$\begin{pmatrix}1&0\\u&1\end{pmatrix} A = \begin{pmatrix}1&a&\frac{1}{2}ua^2\\0&1&ua\\0&0&1\end{pmatrix}$$

The action of this element under Schrödinger represantation:

$$\pi_\hbar\left(\begin{pmatrix}1&0\\u&1\end{pmatrix} A\right): \quad f(x) \mapsto e^{i\hbar\frac{1}{2}ua^2} \cdot e^{iuax} \cdot f(x+\hbar a)$$

The Weil representation of ρ is

$$\rho\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix} : \quad f(x) \longmapsto e^{\frac{-iu}{2\hbar}x^2} f(x)$$

Consider the following commuting diagramm;

$$\begin{tikzcd}
f(x) \arrow[r, mapsto, "\pi_\hbar(A)"] \arrow[d, "\rho\begin{pmatrix}1&0\\u&1\end{pmatrix}"'] & f(x+\hbar a) \arrow[d, "\rho\begin{pmatrix}1&0\\u&1\end{pmatrix}"] \\
 & e^{\alpha x^2}\cdot f(x+\hbar a) \arrow[d, equal] \\
e^{\alpha x^2}\cdot f(x) \arrow[r, "\pi_\hbar\left(\begin{pmatrix}1&0\\u&1\end{pmatrix}A\right)"] & e^{i\hbar\frac{1}{2}ua^2}\cdot e^{iuax}\cdot e^{\alpha(x+\hbar a)^2}\cdot f(x+\hbar a)
\end{tikzcd}$$

here $\alpha = \frac{-iu}{2\hbar}$

A straightforward computation shows that the equality "=" holds and the diagram commutes.

Consider an element $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in H$

The Schrödinger represantation of B acts on $L_2(\mathbb{R})$ by

$$\pi_\hbar(B): \quad f(x) \mapsto e^{ibx} f(x)$$

Our definition of the action of $SL_2(\mathbb{R})$ on H implies:

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

The action of this element under Schrödinger represantation:

$$\pi_\hbar\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} B\right): \quad f(x) \mapsto e^{ibx} f(x)$$

It is easy to see that the following diagram commutes;

$$\begin{tikzcd} f(x) \arrow[r, "\pi_\hbar(B)"] \arrow[d, "\rho\begin{pmatrix}1&0\\u&1\end{pmatrix}"'] & e^{ibx} f(x) \arrow[d, "\rho\begin{pmatrix}1&0\\u&1\end{pmatrix}"] \\ e^{\alpha x^2} f(x) \arrow[r, "\pi_\hbar\left(\begin{pmatrix}1&0\\u&1\end{pmatrix}B\right)"] & e^{ibx} \cdot e^{\alpha x^2} \cdot f(x) \end{tikzcd}$$